DP Analysis & Approaches SL / HL

Maths with Lemon

Lesson 1.6: The Binomial Theorem — Theory Notes

1. What is a Binomial?

A **binomial** is an algebraic expression containing two terms, for example

$$x+2, \qquad 2x-3, \qquad a+b.$$

For small powers we can multiply directly, but the binomial theorem provides an efficient method for expanding $(a+b)^n$.

2. The Binomial Theorem

$$\boxed{(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r}$$

The general term is

$$\boxed{T_{r+1} = \binom{n}{r} a^{n-r} b^r}.$$

Key Idea

The power of a decreases while the power of b increases. In every term, the two powers add to n.

3. Binomial Coefficients

$$\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!}}$$

For example,

$$\binom{5}{2} = \frac{5!}{2!3!} = 10.$$

There are $\boxed{n+1}$ terms in the full expansion of $(a+b)^n$.

4. Fully Expanding a Binomial

Expand $(2+x)^5$:

$$(2+x)^5 = 2^5 + 5(2^4)x + 10(2^3)x^2 + 10(2^2)x^3 + 5(2)x^4 + x^5,$$

so

$$\boxed{(2+x)^5 = 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5}.$$

5. Finding Only the First Few Terms

For $(2+x)^6$,

$$T_{r+1} = \binom{6}{r} 2^{6-r} x^r.$$

Using $r = 0, 1, 2, 3$ gives

$$\boxed{64 + 192x + 240x^2 + 160x^3 + \cdots}.$$

Always check whether the question asks for **increasing** or **decreasing** powers.

6. Finding

a Specified Coefficient

Find the coefficient of y^3 in $(2-3y)^7$.

$$T_{r+1} = \binom{7}{r} 2^{7-r}(-3y)^r.$$

For y^3, $r=3$, so the coefficient is

$$\binom{7}{3} 2^4(-3)^3 = 35(16)(-27) = \boxed{-15120}.$$

7. Multiple x-Terms

For $(x^2+3x)^{15}$,

$$T_{r+1} = \binom{15}{r}(x^2)^{15-r}(3x)^r = \binom{15}{r} 3^r x^{30-r}.$$

To obtain x^{27},

$$30 - r = 27 \Rightarrow r = 3.$$

Thus the coefficient is

$$\boxed{\binom{15}{3} 3^3 = 12285}.$$

8. Negative Powers

For

$$\left(2x^2 - \frac{3}{x}\right)^{10},$$

the general term is

$$\binom{10}{r}(2x^2)^{10-r}\left(-\frac{3}{x}\right)^r = \binom{10}{r} 2^{10-r}(-3)^r x^{20-3r}.$$

The important step is to combine all powers of x before solving for r.

9. Constant Terms

A constant term contains x^0. Therefore, after simplifying the power of x, set

$$\boxed{\text{power of } x = 0}.$$

Then solve for r and substitute that value into the general term.

10. IB-Style Example

For $(3x-2)^{12}$,

$$T_{r+1} = \binom{12}{r}(3x)^{12-r}(-2)^r.$$

For an x^5 term,

$$12 - r = 5 \Rightarrow r = 7.$$

Hence

$$\binom{12}{7}(3x)^5(-2)^7,$$

and the coefficient is

$$\boxed{-24634368}.$$

11. Common Mistakes

- The first term corresponds to $r = 0$, not $r = 1$.
- Term number is $r + 1$.
- Remember $(3x)^r = 3^r x^r$.
- Keep negative signs inside the power, for example $(-2)^r$.
- Combine all powers of x before solving for r.
- For a constant term, set the power of x equal to zero.
- If asked for a coefficient, the final answer must not contain x.

12. Summary

Concept	Rule
Binomial theorem	$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$
General term	$T_{r+1} = \binom{n}{r} a^{n-r} b^r$
Coefficient	$\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$
Number of terms	$n + 1$
Constant term	Set the power of x equal to 0.
Term number	$r + 1$

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Mathematics made clear.